

# Are $R^2$ - and Higgs-inflations really unlikely?

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## Abstract

We address the question of unlikeness of  $R^2$ - and Higgs inflations exhibiting exponentially flat potentials and hence apparently violating the inherent in a chaotic inflation initial condition when kinetic, gradient and potential terms are all of order one in Planck units. Placing the initial conditions in the Jourdan frame we find both models not worse than any other models with unbounded from above potentials: the terms in the Einstein frame are all of the same order, though appropriately smaller.

The  $R^2$ - and Higgs-inflation models [1, 2] work in a very economic way. Inflationary stage is attained due to modification of the gravitational sector with respect to the presently accepted paradigm—General Relativity and Standard Model of particle physics—by introducing a quadratic curvature term in  $R^2$ -inflation and a strong nonminimal interaction with the Higgs boson in Higgs-inflation. In the Einstein frame, where the gravity takes the Einstein–Hilbert form, this stage is realized as a slow-roll inflation at super-Planckian values of a corresponding scalar degree of freedom in each theory. The scalar has very flat plateau-like potential at large values of the field bounded from above by the value of

$$V_0 \simeq 10^{-12} \times M_{\text{Pl}}^4, \quad (1)$$

which comes from fitting to the matter power spectrum extracted from the CMB anisotropy map [3, 4] and observations of the Large Scale Structure [5, 6].

In this work we address the issues of formulated in Ref. [7] “unlikeness” problem to the  $R^2$ -inflation and Higgs-inflation. This problem originates in the initial conditions for a

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successful inflation. According to Refs. [8, 9] for a chaotic inflation to begin in a relatively uniform domain of the Planckian size it is sufficient to have *each* of the forms of inflaton energy density being of the Planckian value, i.e.<sup>1</sup>

$$\frac{1}{2}\dot{\phi}^2 \sim \frac{1}{2}(\partial_i\phi)^2 \sim V(\phi) \sim M_{\text{Pl}}^4. \quad (2)$$

However, for the models with plateau-like potential,  $V(\phi) \simeq V_0$ , the initial condition one might expect is

$$\frac{1}{2}\dot{\phi}^2 \sim \frac{1}{2}(\partial_i\phi)^2 \sim M_{\text{Pl}}^4 \gg V(\phi).$$

In this case the kinetic and gradient inflaton energy densities quickly dominate and hinder the inflation from launching. The Universe starting with these initial conditions undergoes inflationary expansion later, only if the initial uniform space region would be much bigger than the Planckian domain. Such initial homogeneity of the Universe looks very unnatural and “unlikely”.

We start consideration of this problem (see e.g. [10, 11] for earlier studies) with  $R^2$ -inflation. Lagrangian of this theory in the Jordan frame (JF) is<sup>2</sup>

$$S^{JF} = -\frac{M_{\text{Pl}}^2}{16\pi} \int \sqrt{-g^{JF}} d^4x \left[ R^{JF} - \frac{(R^{JF})^2}{6\mu^2} \right]. \quad (3)$$

It is convenient to go to the Einstein frame (EF) where the gravity action takes the Einstein-Hilbert form and the scalar degree of freedom responsible for the inflationary solution becomes canonically normalized. For this purpose we introduce a new auxiliary scalar field  $Q$  and write the action (3) in the following way [13]

$$S^{JF} = -\frac{M_{\text{Pl}}^2}{16\pi} \int \sqrt{-g^{JF}} d^4x \left[ \left(1 - \frac{Q}{3\mu^2}\right) (R^{JF} - Q) + \left(Q - \frac{Q^2}{6\mu^2}\right) \right]. \quad (4)$$

Varying (4) with respect to  $Q$  we obtain

$$Q = R^{JF}, \quad (5)$$

and eq. (4) reduces to original action (3). Then we get rid of the factor

$$\Omega^2 = 1 - \frac{Q}{3\mu^2} \quad (6)$$

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<sup>1</sup>To simplify the formulas hereinafter, we omit the scale factor; it can be straightforwardly recovered depending on the choice of metric.

<sup>2</sup>Following [12] we choose the Landau-Lifshitz convention for the variables in the gravity sector, so that  $(+, -, -, -)$  is the metric signature, and the scalar curvature is negative at inflationary stage.

by proper rescaling, that is by making conformal transformation of the metric

$$g_{\mu\nu}^{EF} = \Omega^2 g_{\mu\nu}^{JF}, \quad \text{with} \quad \Omega^2 \equiv \exp\left(\sqrt{\frac{\pi}{3}} \frac{4\phi}{M_{\text{Pl}}}\right). \quad (7)$$

Thus we arrive at the following action in the EF

$$S^{EF} = \int \sqrt{-g^{EF}} d^4x \left[ -\frac{M_{\text{Pl}}^2}{16\pi} R^{EF} + \frac{1}{2} g_{\mu\nu}^{EF} \partial^\mu \phi \partial^\nu \phi - \frac{3\mu^2 M_{\text{Pl}}^2}{32\pi} \left(1 - \frac{1}{\Omega^2}\right)^2 \right]. \quad (8)$$

The homogeneous and isotropic Universe described by the action (3) undergoes an inflationary expansion at large values of  $R^{JF}$ . In the EF this stage is realized as slow-roll inflation taking place at large values of the scalar field,  $\phi \gg M_{\text{Pl}}$  which plays the role of the inflaton. Normalization of the amplitude of the generated during inflation scalar perturbations to the spectra of observed CMB anisotropy and Large Scale Structure (1) yields the estimate  $\mu \approx 2.5 \times 10^{-6} \times M_{\text{Pl}}$ .

Now let us discuss initial conditions for this model. Looking at the action in the EF (8) one may conclude that the theory suffers from "unlikeness" problem raised in Ref. [7]. Indeed, at large  $\phi$  the potential is very flat and tends to the constant (1),  $V \rightarrow V_0 = 3\mu^2 M_{\text{Pl}}^2 / (32\pi) \simeq 10^{-12} \times M_{\text{Pl}}^4$ . To clarify this question let us formulate a natural initial condition in the JF which is the physical frame of the theory. In this frame the model is described by pure gravitational action (3), thus in the sense of Refs. [8, 9] one expects that the Universe in a Planck scale domain has the Planckian curvature

$$|R^{JF}| \sim M_{\text{Pl}}^2. \quad (9)$$

According to eqs. (5), (6), for the transformation function  $\Omega$  connecting the two frames we have

$$\Omega \sim M_{\text{Pl}}/\mu \sim 10^6. \quad (10)$$

This leads to  $\phi \sim |\log(M_{\text{Pl}}^2/\mu^2)| \sim 20$  for the initial inflaton field value and  $V(\phi) = V_0 \simeq 10^{-12} \times M_{\text{Pl}}^4$  for the initial potential energy, as required, see (1).

In order to estimate the initial kinetic and gradient energy of the inflaton in the Einstein frame one may use the relation

$$-R^{JF} = -\Omega^2 R^{EF} + \Omega^2 \frac{8\pi}{M_{\text{Pl}}^2} g_{\mu\nu}^{EF} \partial^\mu \phi \partial^\nu \phi - \Omega^2 \frac{4\sqrt{3\pi}}{M_{\text{Pl}}} g_{\mu\nu}^{EF} \partial^\mu \partial^\nu \phi, \quad (11)$$

between Ricci scalars in the two frames. The first two terms in r.h.s. of eq. (11) refer to the pure gravity and scalaron, whose initial contributions to the the energy density are of the

same order, according to Refs. [8, 9] and equations of motion in the EF. Hence

$$|R^{JF}| \Omega^{-2} \sim |R^{EF}| \sim \dot{\phi}^2 M_{\text{Pl}}^{-2} \sim (\partial_i \phi)^2 M_{\text{Pl}}^{-2}.$$

Recalling (9) we get  $\frac{1}{2}\dot{\phi}^2 \sim \frac{1}{2}(\partial_i \phi)^2 \sim M_{\text{Pl}}^4 \Omega^{-2}$ . Then taking the potential  $V(\phi)$  from (8) and inserting the estimate (10) we obtain finally

$$\frac{1}{2}\dot{\phi}^2 \sim \frac{1}{2}(\partial_i \phi)^2 \sim V(\phi) \sim \mu^2 M_{\text{Pl}}^2 / 30 \sim 10^{-12} \times M_{\text{Pl}}^4.$$

But it is just what is needed for a successfully inflationary model, see eq. (1). We observe, that in the Einstein frame all the relevant terms start with the same initial value, which is however smaller, than the Planck mass.

One can find the very similar arguments working well for the Higgs-inflation, which is not worse than  $R^2$ -inflation in this respect. The action of the Higgs-inflation model in the JF is [2]

$$S^{JF} = \int \sqrt{-g^{JF}} d^4x \left[ -\frac{M_{\text{Pl}}^2}{16\pi} \left( 1 + \frac{8\pi\xi h^2}{M_{\text{Pl}}^2} \right) R^{JF} + \frac{1}{2} g_{\mu\nu}^{JF} \partial^\mu h \partial^\nu h - \frac{\lambda}{4} h^4 \right], \quad (12)$$

where we use the unitary gauge with  $h$  being the Higgs boson. Going to the EF by conformal transformation (7) with

$$\Omega^2 = 1 + \frac{8\pi\xi h^2}{M_{\text{Pl}}^2} \quad (13)$$

we arrive at the following action in the EF

$$S^{EF} = \int \sqrt{-g^{EF}} d^4x \left[ -\frac{M_{\text{Pl}}^2}{16\pi} R^{EF} + \frac{1}{2} g_{\mu\nu}^{EF} \partial^\mu \phi \partial^\nu \phi - \frac{\lambda}{4} \frac{h(\phi)^4}{\Omega(\phi)^4} \right], \quad (14)$$

where we replace  $h$  with canonically normalized scalar field  $\phi$  utilizing the relation

$$\frac{d\phi}{dh} = \sqrt{\frac{\Omega^2 + 48\pi\xi^2 h^2 / M_{\text{Pl}}^2}{\Omega^4}}. \quad (15)$$

Inflation in this model in the JF happens at large values of  $h \gg M_{\text{Pl}}/8\pi\sqrt{\xi}$ . In this limit from eq. (15) we get

$$h \simeq \frac{M_{\text{Pl}}}{\sqrt{8\pi\xi}} \exp \left( 2\sqrt{\frac{\pi}{3}} \frac{\phi}{M_{\text{Pl}}} \right). \quad (16)$$

In this case the inflaton potential in the EF becomes exponentially flat and takes form (cf. eq. (8))

$$\frac{\lambda}{4} \frac{h(\phi)^4}{\Omega(\phi)^4} \simeq \frac{\lambda M_{\text{Pl}}^4}{256 \pi^2 \xi^2} \left[ 1 + \exp \left( -4\sqrt{\frac{\pi}{3}} \frac{\phi}{M_{\text{Pl}}} \right) \right]^2. \quad (17)$$

Normalization of the amplitude of the scalar perturbations generated in this model during inflation yields the estimate

$$\xi \simeq 47000\sqrt{\lambda} \sim 1.5 \times 10^4 . \quad (18)$$

This implies

$$V_0 \simeq 1.8 \times 10^{-13} \times M_{\text{Pl}}^4 \quad (19)$$

for the upper bound.

We start the discussion with initial conditions in the JF (12). In the sense of Refs. [8, 9] (see Ref. [14] for an alternative approach) one may expect that the Universe starts with near Planckian values of the energy density in all the species. For the potential energy we immediately get

$$\lambda h^4/4 \sim M_{\text{Pl}}^4 , \quad (20)$$

and hence  $h \sim M_{\text{Pl}}$ . It corresponds to the plateau-like part of the scalar field potential in the EF,  $V(\phi) \sim V_0 \simeq 10^{-13} \times M_{\text{Pl}}^4$  and  $\Omega^2 \sim 5 \times 10^6$ . But with the kinetic energy density the situation is not so clear due to nonminimal coupling to gravity in (12). As soon as Ricci scalar  $R^{JF}$  contains derivatives, this term, proportional to  $\xi$  (18), represents large mixing between kinetic term of the scalar field and metric derivatives. One can make it explicit, for example, by expanding the curvature above the Minkowski background. From eqs. (15) and (18) one observes that this term gives the main contribution to the kinetic part of the action for  $\phi$  in the EF.

Moreover, a non-zero value of  $h$  rescales the gravity mass scale:  $\Omega M_{\text{Pl}} \rightarrow M_{\text{Pl}}$ , see (12), (13). Then, following Refs. [8, 9], the gravity contribution to the total energy density, the scalar kinetic terms and the scalar potential are expected to be of the same order, which yields the reliable initial conditions

$$\Omega^2 M_{\text{Pl}}^2 R^{JF} \sim \xi \dot{h}^2 \sim \xi (\partial_i h)^2 \sim \lambda h^4 . \quad (21)$$

Adopting the estimate (20) we obtain then

$$R^{JF} \sim M_{\text{Pl}}^2/\Omega^2 \quad (22)$$

consistently with the equation of motions in the JF. Using the relation between the Ricci scalars in the two frames, which in the case of large  $\Omega$  coincides with eq. (11), we arrive at the following initial conditions in the EF

$$\frac{1}{2}\dot{\phi}^2 \sim \frac{1}{2}(\partial_i \phi)^2 \sim M_{\text{Pl}}^4/\Omega^4 \sim 10^{-13} M_{\text{Pl}}^4 ,$$

that is right what we need, eq. (19).

We emphasize that in each frame all the terms, when correctly identified, are of the same order, but the scales in the JF and in the EF are different, which reminds the situation in  $R^2$ -inflation. In the Higgs-inflation, since at large  $\Omega$  the value of  $\sqrt{8\pi\xi}$  replaces the Planck scale in (12), one can introduce any value  $\Lambda$  to be utilized instead of  $M_{\text{Pl}}$  in eq. (20) and to be accepted as a universal initial condition in the JF (thus all the terms in (21) are of order  $\Lambda^4$ ). Then  $R^{JF} \sim \Lambda^4 \Omega^{-2} M_{\text{Pl}}^{-2}$  supplants eq. (22) and the reliable initial condition in the EF (all terms there are always of the same order) is obtained with  $\Omega \sim \sqrt{\xi} \Lambda M_{\text{Pl}}^{-1}$ , as follows from (13).

To summarize, one concludes that for  $R^2$ - and Higgs-inflations formulated in the JF all species of the initial inflaton energy density should be of the same order. Then the counterparts in the EF are all of the same order as well. In this sense these models of inflation are not as “unlikely” as other models with unbounded potentials. An improper for the successful inflation initial condition with a hierarchy between the terms in the EF (e.g. when gradients dominate) would imply the same hierarchy in the JF, which is unnatural.

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